

Indirect Proofs

Outline for Today

- ***What is an Implication?***
 - Understanding a key type of mathematical statement.
- ***Negations and their Applications***
 - How do you show something is *not* true?
- ***Proof by Contrapositive***
 - What's a contrapositive?
 - And some applications!
- ***Proof by Contradiction***
 - The basic method.
 - And some applications!

Logical Implication

Implications

- An ***implication*** is a statement of the form

If P is true, then Q is true.

- Some examples:
 - If n is an even integer, then n^2 is an even integer.
 - If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
 - If you like the way you look that much, (ohhh baby) then you should go and love yourself.

Implications

- An ***implication*** is a statement of the form

If P is true, then Q is true.

- In the above implication, the statement “ P is true” is called the ***antecedent*** and the statement “ Q is true” is called the ***consequent***.

What Implications Mean

- Consider the simple statement

If I put fire near cotton, it will burn.

- Some questions to consider:
 - Does this apply to all fire and all cotton, or just some types of fire and some types of cotton? (*Scope*)
 - Does the fire cause the cotton to burn, or does the cotton burn for another reason? (*Causality*)
- These are significantly deeper questions than they might seem.
- To mathematically study implications, we need to formalize what implications really mean.

Understanding Implications

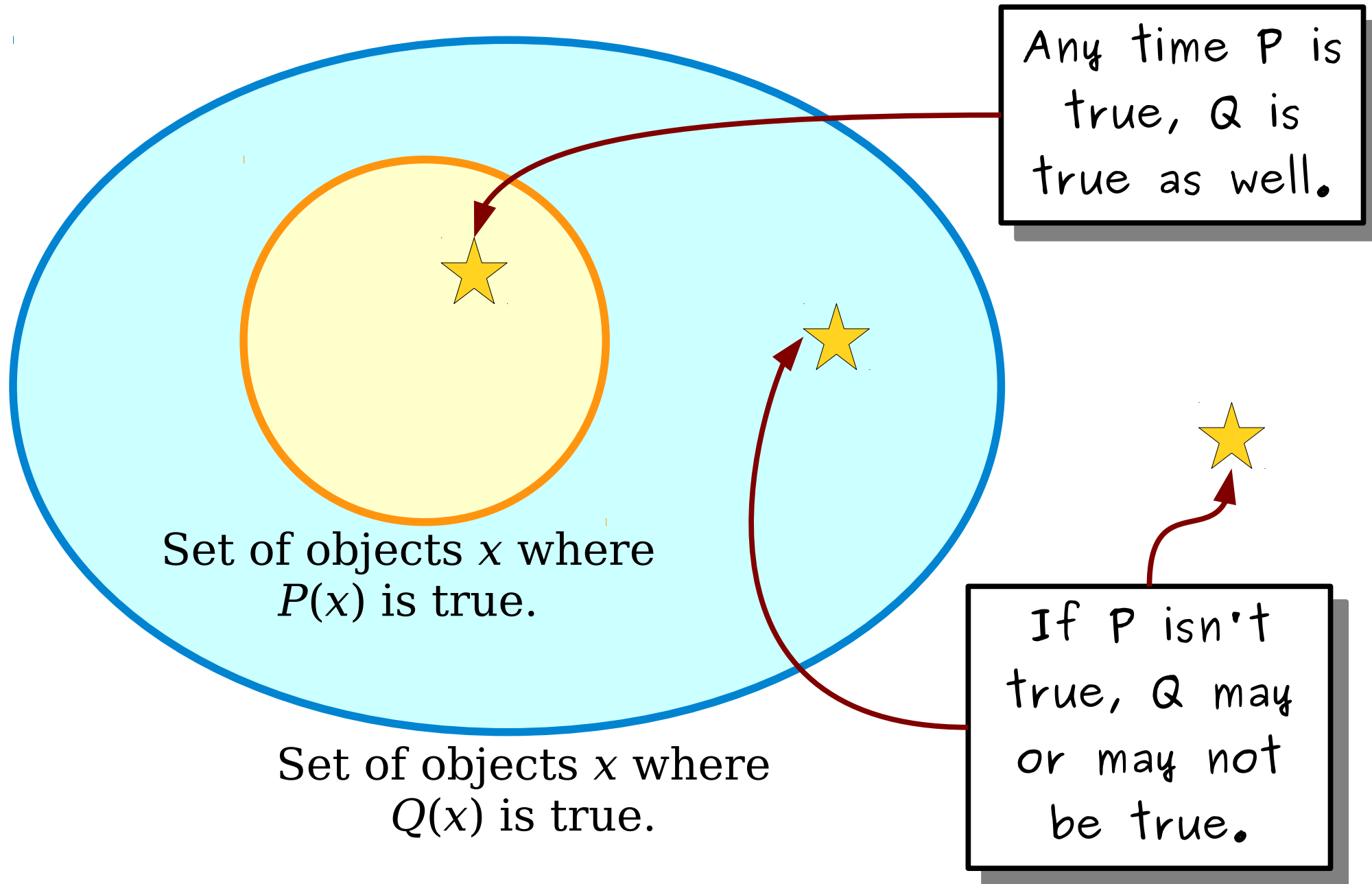
**“If there's a rainbow in the sky,
then it's raining somewhere.”**

- In mathematics, implication is *directional*.
 - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
 - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about *causality*.
 - Rainbows do not cause rain. ☺

What Implications Mean

- In mathematics, a statement of the form **For any x , if $P(x)$ is true, then $Q(x)$ is true** means that any time you find an object x where $P(x)$ is true, you will see that $Q(x)$ is also true (for that same x).
- There is no discussion of causation here. It simply means that if you find that $P(x)$ is true, you'll find that $Q(x)$ is also true.

Implication, Diagrammatically



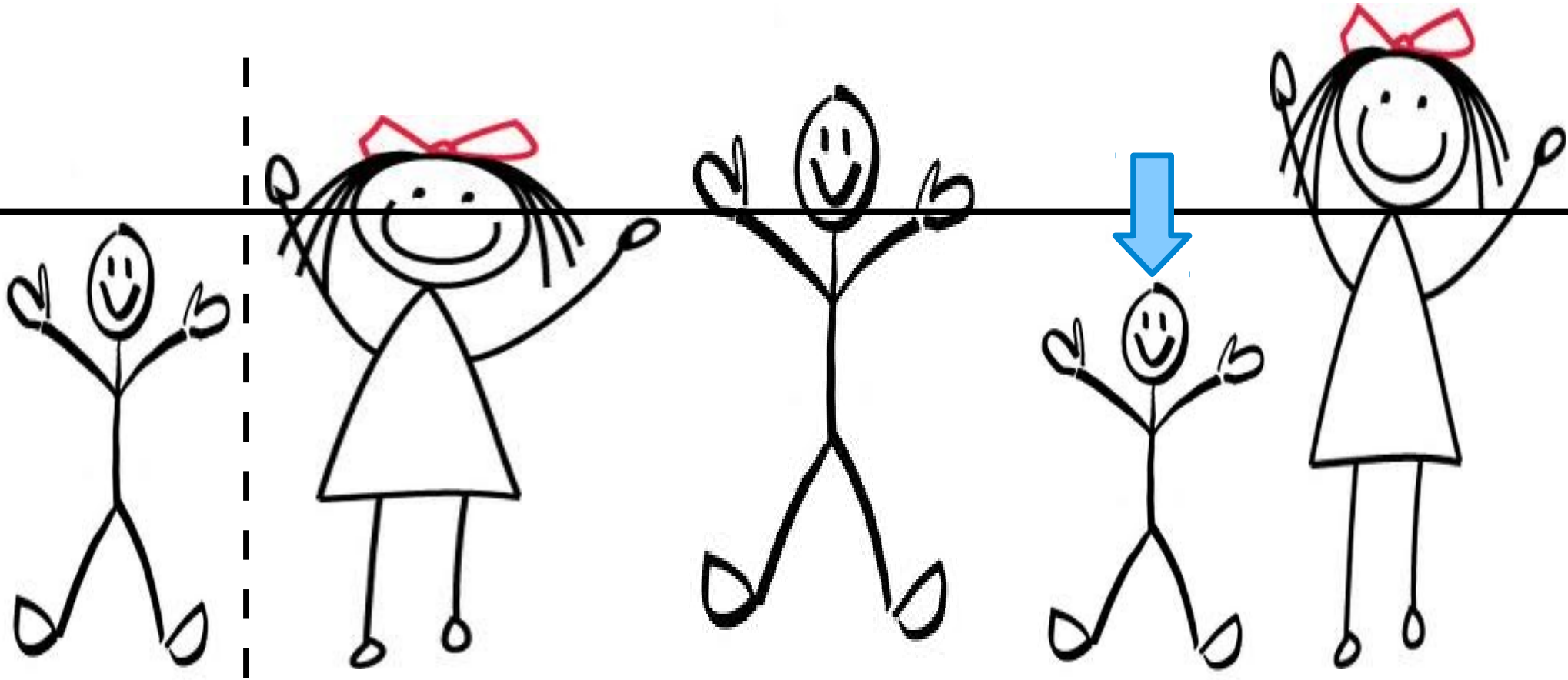
Negations

Negations

- A **proposition** is a statement that is either true or false.
- Some examples:
 - If n is an even integer, then n^2 is an even integer.
 - $\emptyset = \mathbb{R}$.
 - The new me is still the real me.
- The **negation** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.
- For example, consider the statement “it is snowing outside.”
 - Its negation is “it is not snowing outside.”
 - Its negation is *not* “it is sunny outside.” ⚠
 - Its negation is *not* “we’re in the Bay Area.” ⚠

How do you find the negation
of a statement?

“All My Friends Are Taller Than Me”



Me

My Friends

The negation of the *universal* statement

Every P is a Q

is the *existential* statement

There is a P that is not a Q .

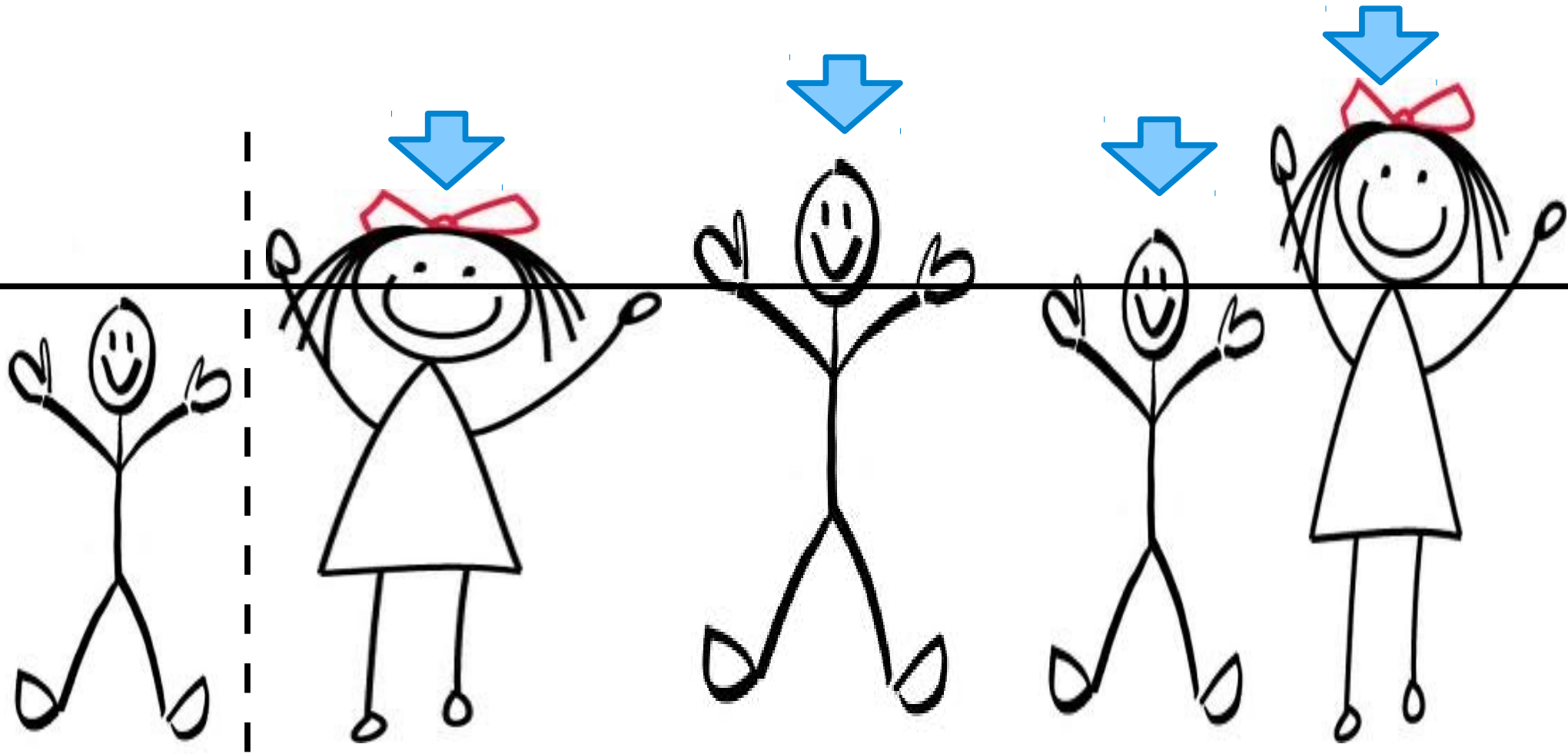
The negation of the *universal* statement

For all x , $P(x)$ is true.

is the *existential* statement

There exists an x where $P(x)$ is false.

“Some Friend Is Shorter Than Me”



Me

My Friends

The negation of the *existential* statement

There exists a P that is a Q

is the *universal* statement

Every P is not a Q .

The negation of the *existential* statement

There exists an x where $P(x)$ is true

is the *universal* statement

For all x , $P(x)$ is false.

Puppy Logic

- Consider the statement

If x is a puppy, then I love x .

- The statement below is ***not*** its negation:

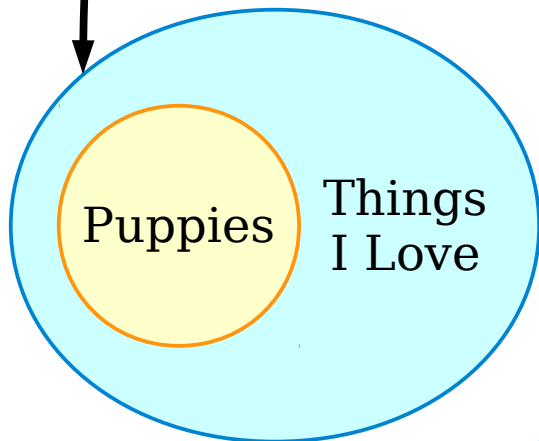
⚠ If x is a puppy, then I don't love x ⚠

- It might not be obvious, at a first glance, why these statements aren't negations of one another. Talk this over with someone next to you and see if you can sort out why.

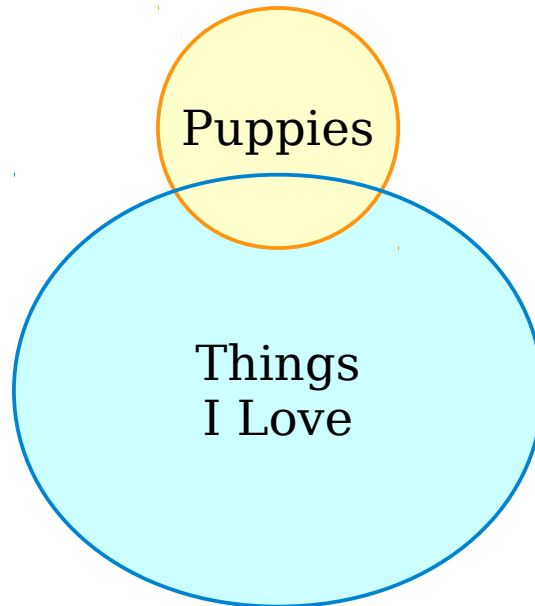
Puppy Logic

If x is a puppy, then I love x .

⚠ If x is a puppy, then I don't love x ⚠

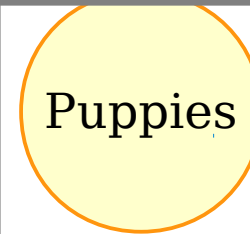


"I love *all* puppies."



"It's complicated."

So what is the proper negation?

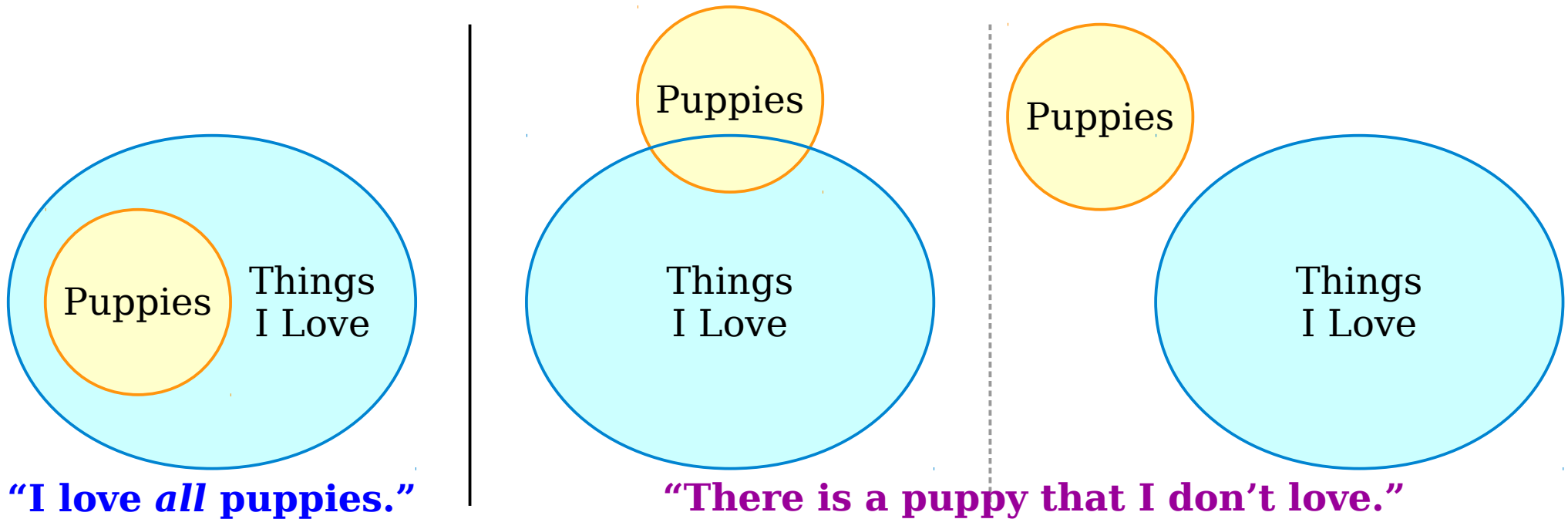


"I don't love *any* puppies."

Puppy Logic

If x is a puppy, then I love x .

There is a puppy I don't love.



The negation of the statement

**“For any x , if $P(x)$ is true,
then $Q(x)$ is true”**

is the statement

**“There is at least one x where
 $P(x)$ is true and $Q(x)$ is false.”**

***The negation of an implication
is not an implication!***

How to Negate Universal Statements:

“For all x , $P(x)$ is true”

becomes

“There is an x where $P(x)$ is false.”

How to Negate Existential Statements:

“There exists an x where $P(x)$ is true”

becomes

“For all x , $P(x)$ is false.”

How to Negate Implications:

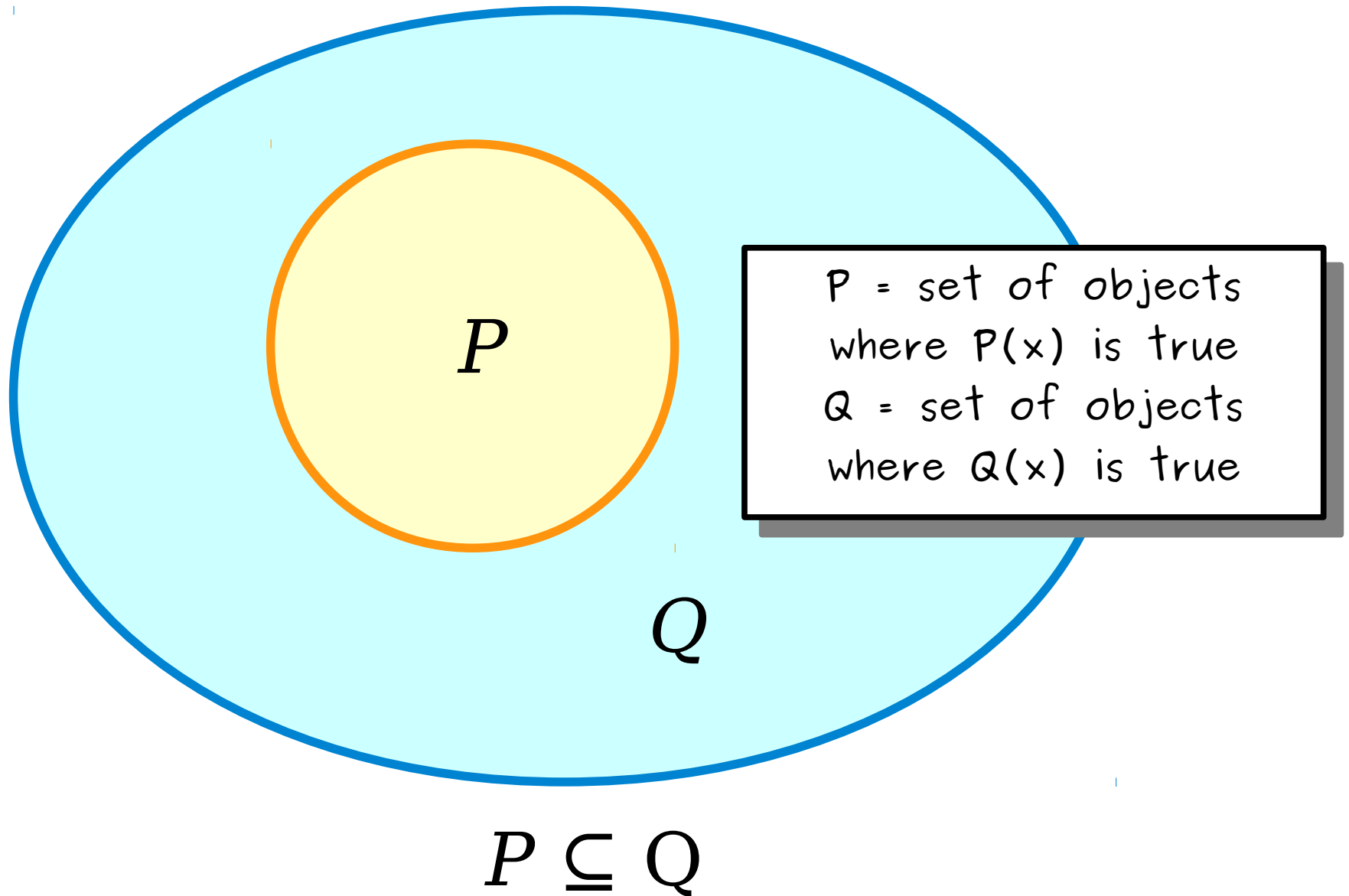
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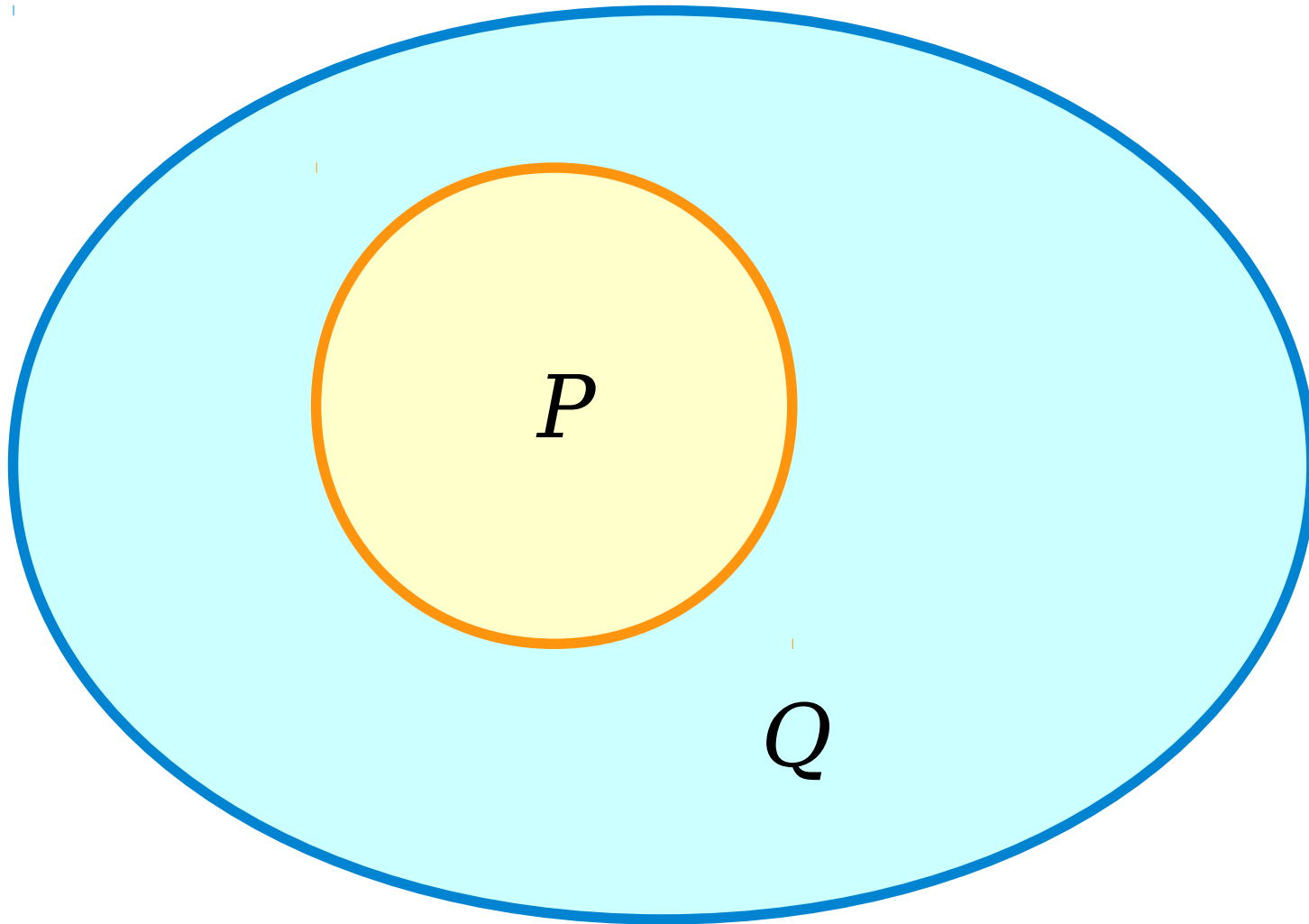
“There is an x where $P(x)$ is true and $Q(x)$ is false”

A Different Perspective on Implication

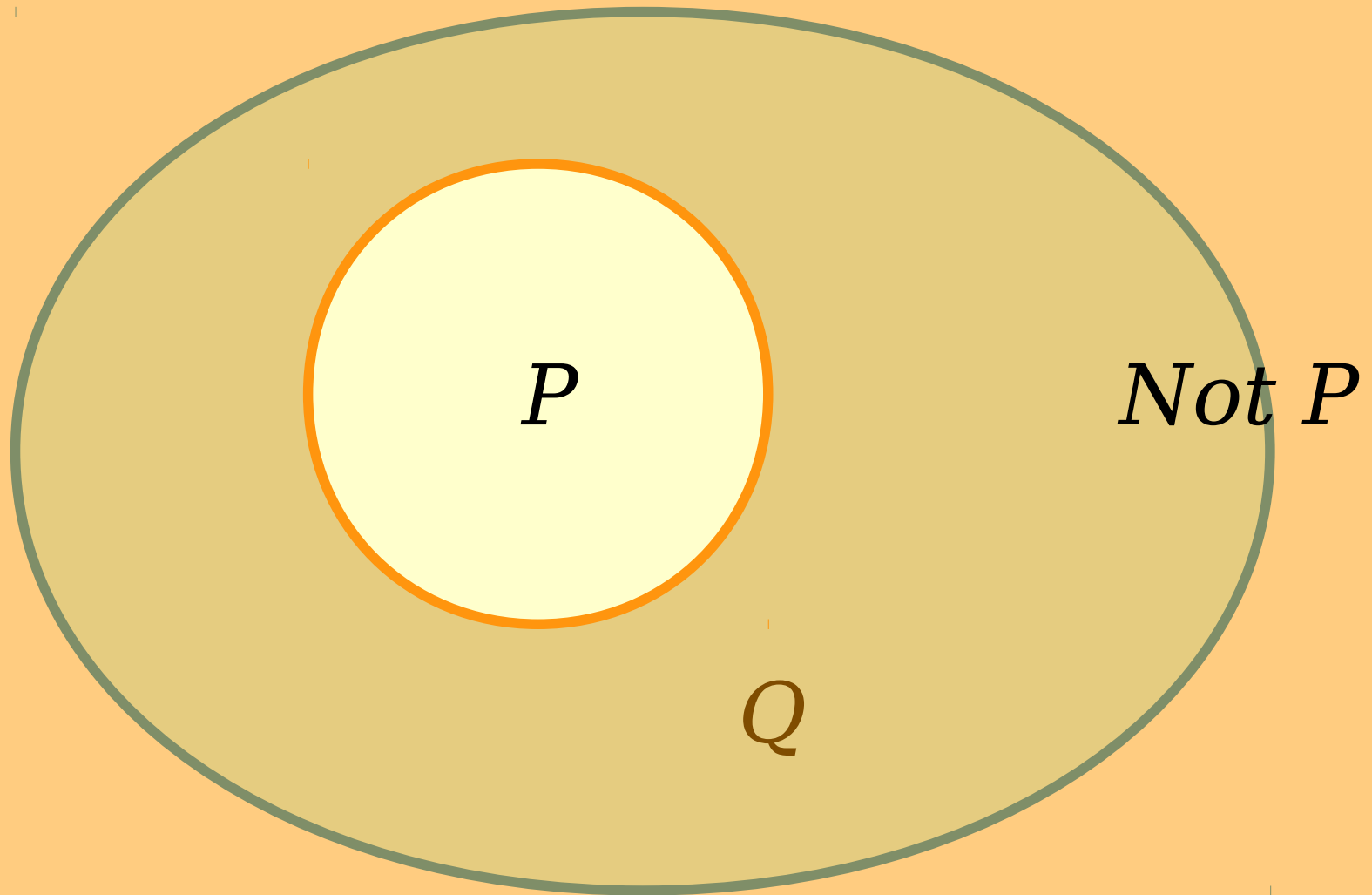
Implication, Diagrammatically



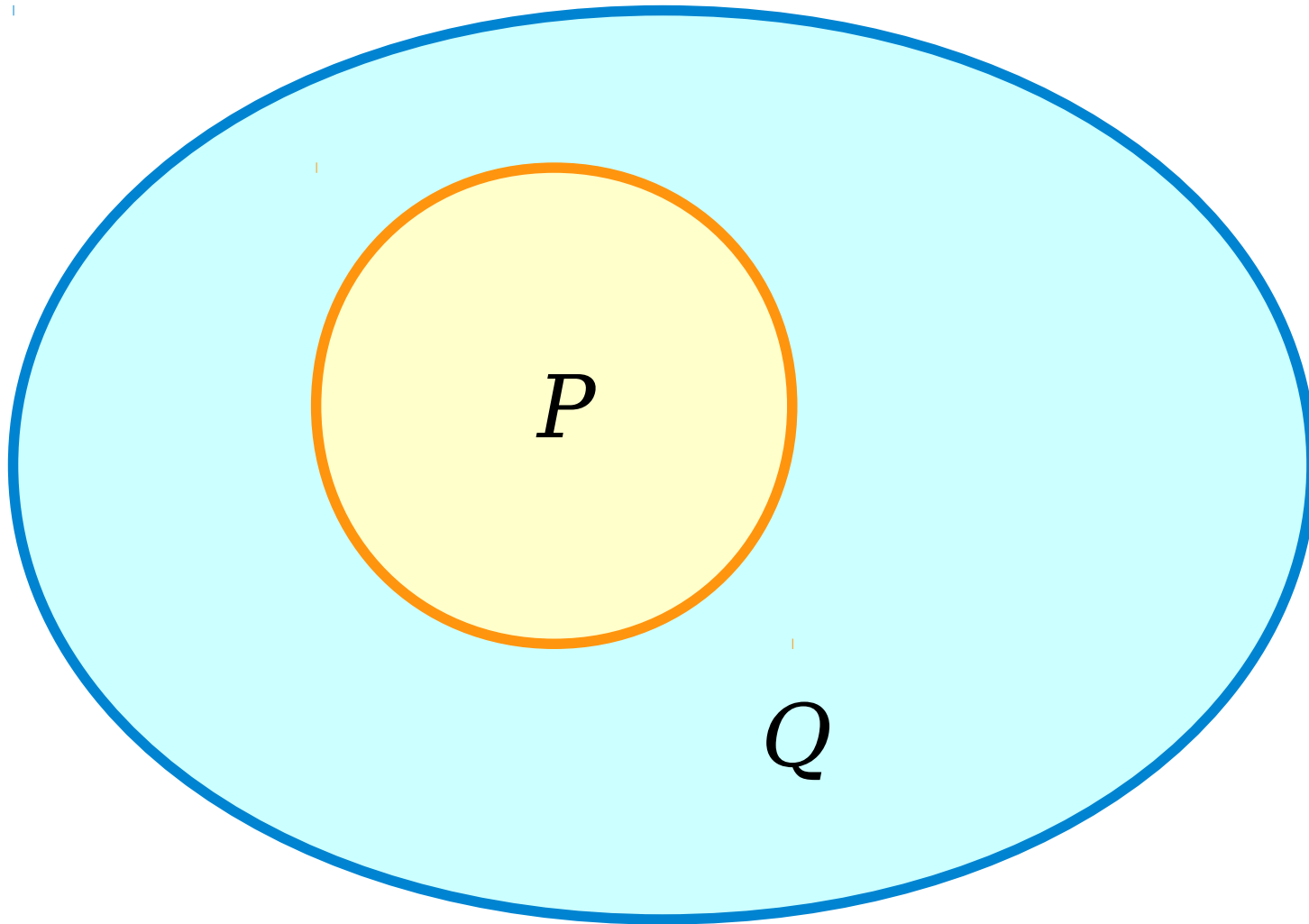
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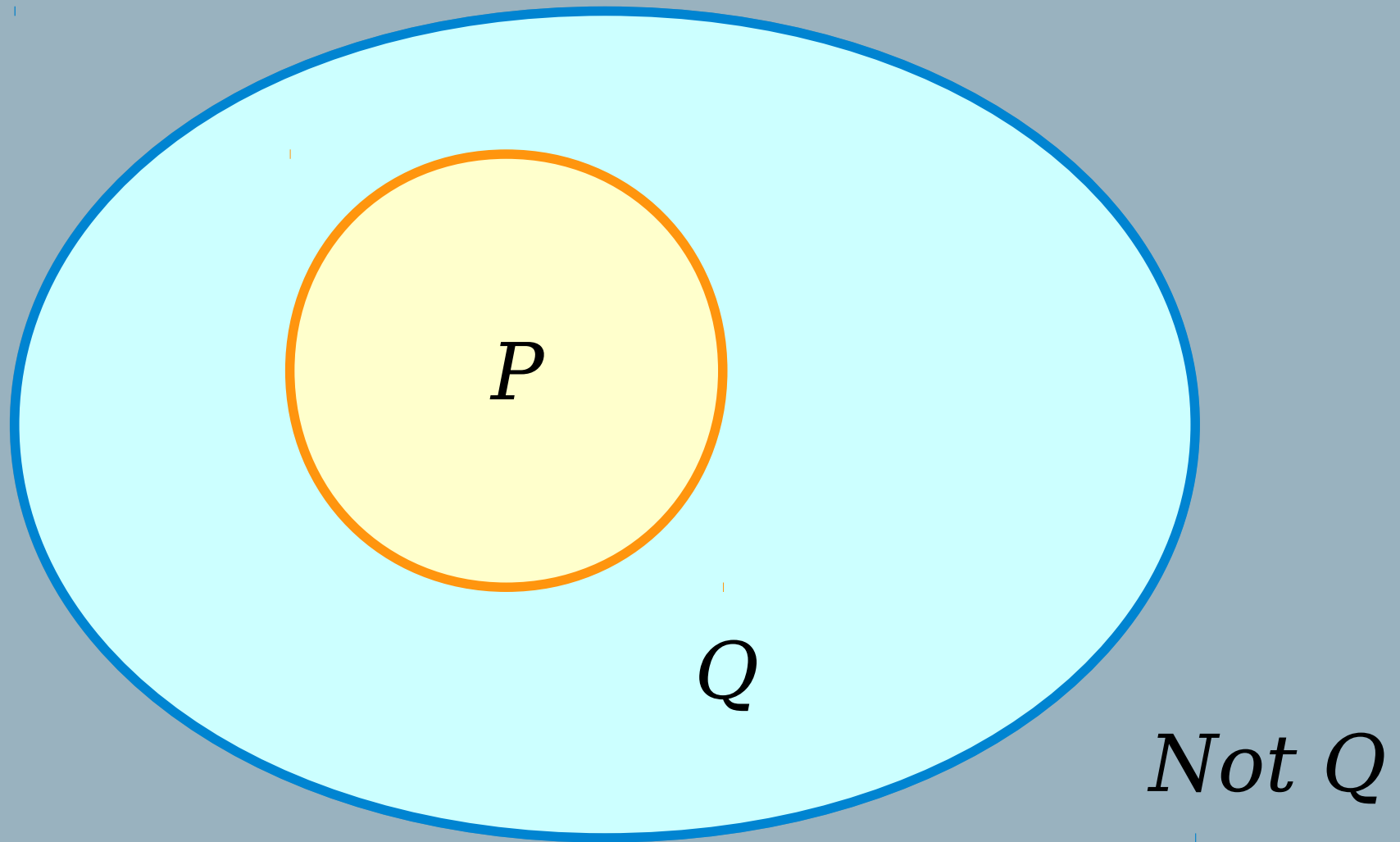
Implication, Diagrammatically



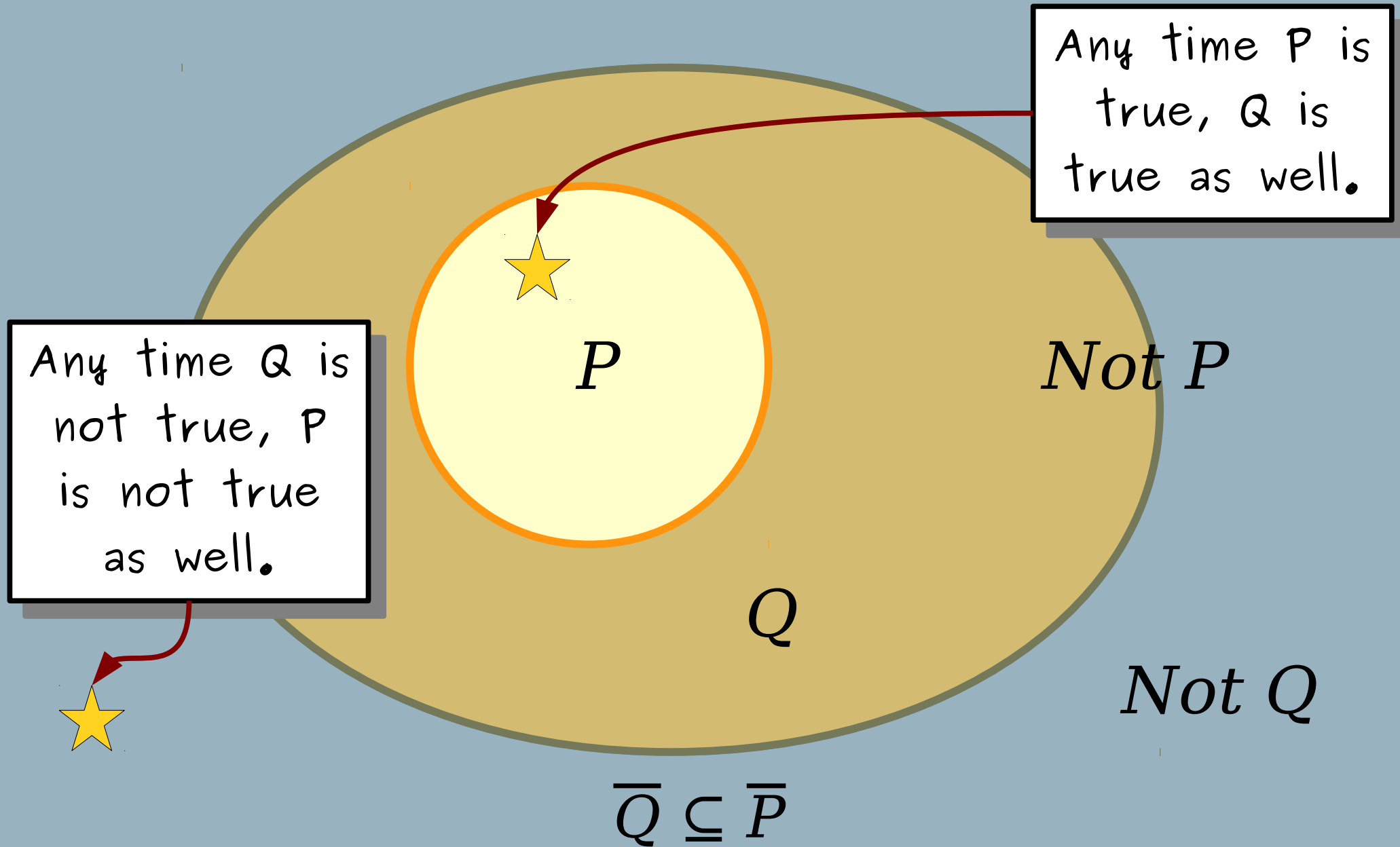
Implication, Diagrammatically



Implication, Diagrammatically



Implication, Diagrammatically



Proof by Contrapositive

The Contrapositive

- The **contrapositive** of the implication “If P , then Q ” is the implication “If Q is false, then P is false.”
- For example:
 - “If it’s a puppy, then I love it.”
 - Contrapositive: “If I don’t love it, then it’s not a puppy.”
- Another example:
 - “If I store cat food inside, then angry raccoons won’t steal my cat food.”
 - Contrapositive: “If angry raccoons stole my cat food, then I didn't store it inside.”

To prove the statement

“if **P is true**, then **Q is true**,”

you can choose to instead prove the
equivalent statement

“if **Q is false**, then **P is false**,”

if that seems easier.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof:

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This is a courtesy to the reader and says "heads up! we're not going to do a regular old-fashioned direct proof here."

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove this by contrapositive

What is the contrapositive of the
statement

if n^2 is even, then n is even?

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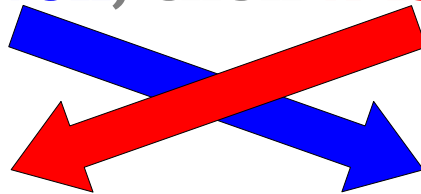
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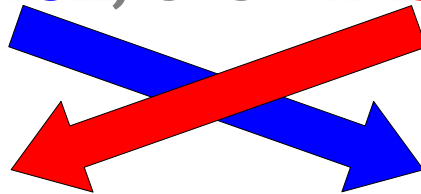


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What is the contrapositive of the statement

if n^2 is even, then n is even?



If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove this by contrapositive by showing that if n is odd, then n^2 is odd.

What is the contrapositive of the statement

if n^2 is even, then n is even?

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Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

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We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove this by contrapositive by showing that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove this by contrapositive by showing that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove this by contrapositive by showing that if n is odd, then n^2 is odd.

Let n be an arbitrary odd integer. Since n is odd, there is some integer k such that $n = 2k + 1$.

Squaring both sides of this equality and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

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Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

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Squaring both sides of this equality and simplifying gives the following:

$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

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Therefore, n^2 is odd.

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$$n^2 = 4k^2 + 4k + 1$$

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Proof: We will prove this by contrapositive by showing that if n is odd, then n^2 is odd.

Let n be a
there is some
Squaring
simplifying

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.
2. Explicitly state the contrapositive of what we want to prove.
3. Go prove the contrapositive.

From this
(namely, 2
Therefore

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. Therefore, n^2 is odd. ■

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

For any integer n , if n is even, then n^2 is even.

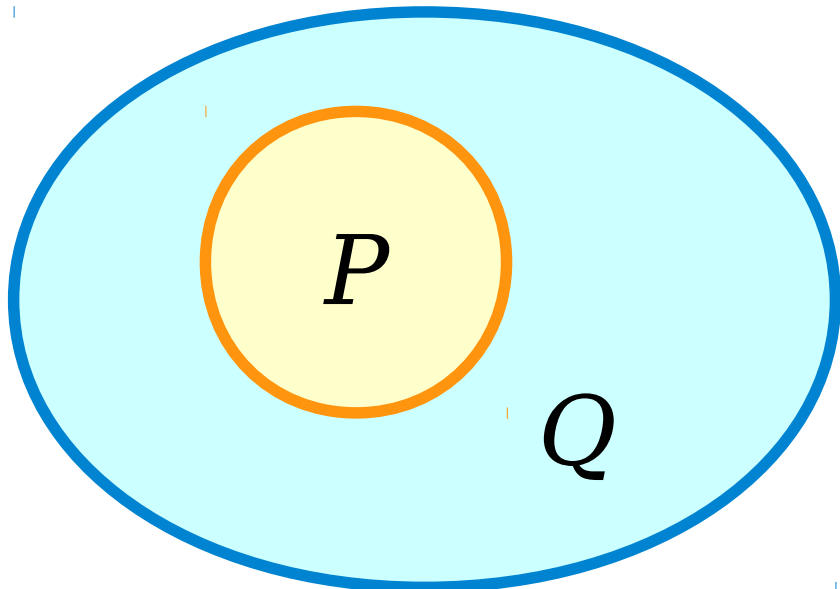
For any integer n , if n^2 is even, then n is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say
for any integer n : n is even if and only if n^2 is even.

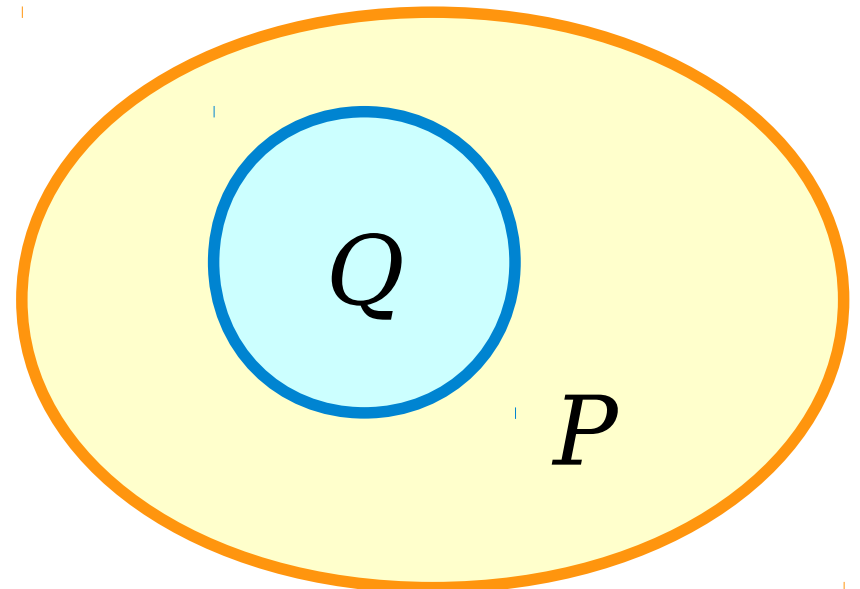
Proving Biconditionals

- To prove a theorem of the form
 P if and only if Q ,
you need to prove two separate statements.
 - First, that if P is true, then Q is true.
 - Second, that if Q is true, then P is true.
- You can use any proof techniques you'd like to show each of these statements.
 - In our case, we used a direct proof for one and a proof by contrapositive for the other.

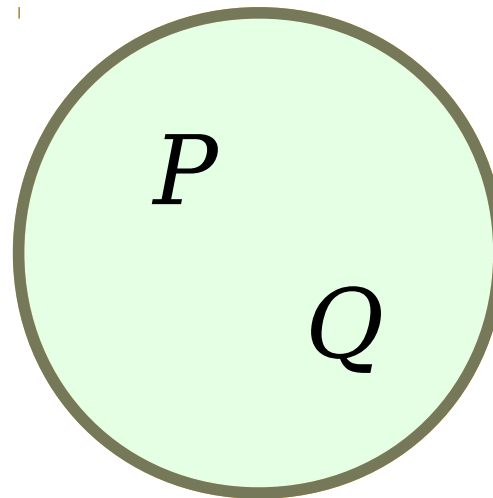
Biconditionals, Diagrammatically



If P, then Q



If Q, then P



P if and only if Q

Time-Out for Announcements!

Handouts

- There are *five* total handouts released today:
 - Mathematical Vocabulary
 - Guide to Indirect Proofs
 - Ten Techniques to Get Unstuck
 - Proofwriting Checklist
 - ***Problem Set One***
- Be sure to read over these; there's a lot of really important information in there!

Announcements

- Problem Set 1 goes out today!
- **Checkpoint** due Monday, July 1 at 3:00PM.
 - Grade determined by attempt rather than accuracy. It's okay to make mistakes – we want you to give it your best effort, even if you're not completely sure what you have is correct.
 - We will get feedback back to you with comments on your proof technique and style.
 - The more effort you put in, the more you'll get out.
- **Remaining problems** due Friday, July 5 at 3:00PM.
 - Feel free to email us with questions, stop by office hours, or ask questions on Piazza!

Submitting Assignments

- All assignments should be submitted through GradeScope.
 - The programming portion of the assignment gets submitted separately from the written component.
 - The written component **must** be typed up; handwritten solutions don't scan well and get mangled in GradeScope.
- Summary of the late policy:
 - Everyone has *three* 24-hour late days.
 - Late days can't be used on checkpoints.
 - Nothing may be submitted more than **two** days past the due date.
- Because submission times are recorded automatically, we're strict about the submission deadlines.
- **Very good idea:** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
- **Very bad idea:** Wait until the last minute to submit.

Working in Pairs

- You can work on the problem sets individually or in pairs.
- Each person/pair should only submit a single problem set. In other words, if you're working in a pair, you and your partner should agree who will make the submission.
- Full details about the problem sets, collaboration policy, and Honor Code can be found in Handout 04 and Handout 05.

A Note on the Honor Code

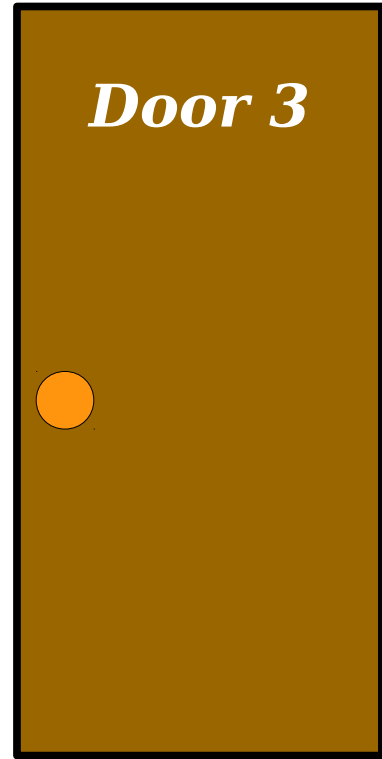
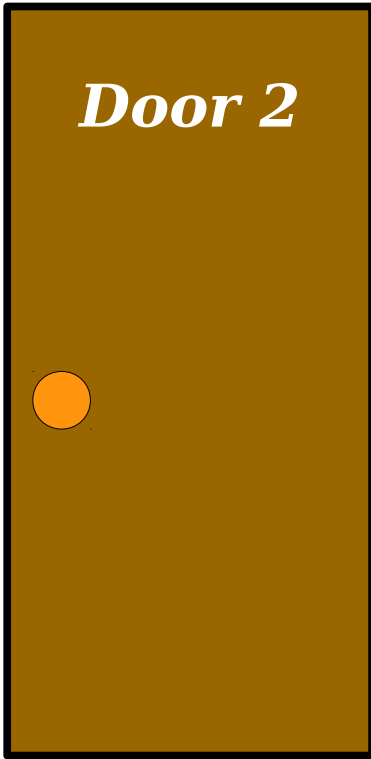
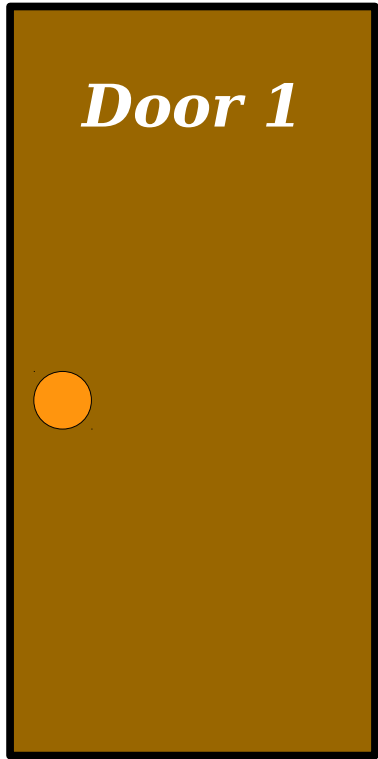
Office hours have started!

Schedule is available
on the course website.

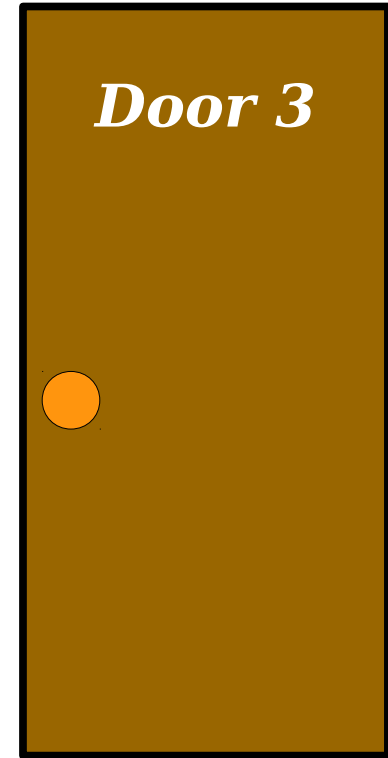
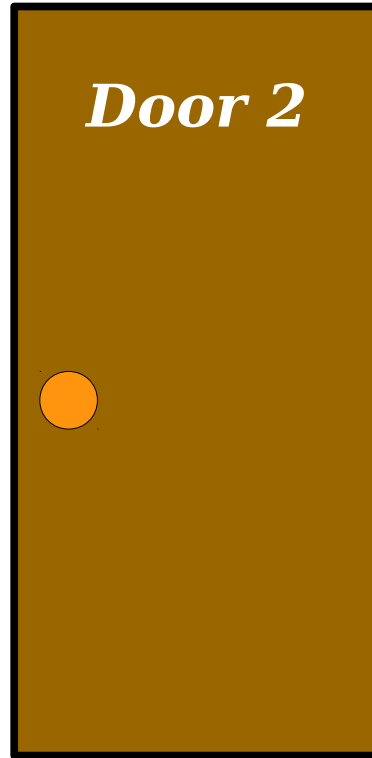
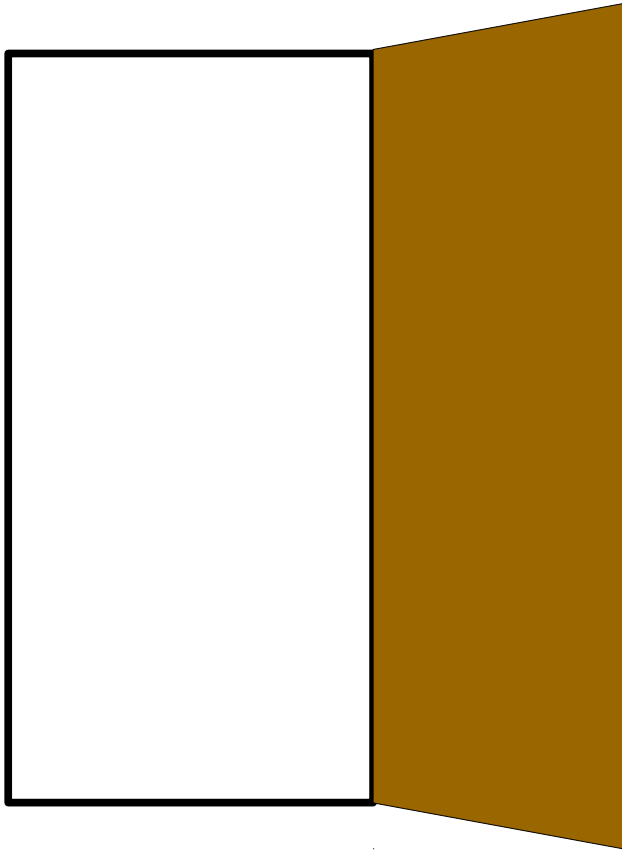
Back to CS103!

Proof by Contradiction

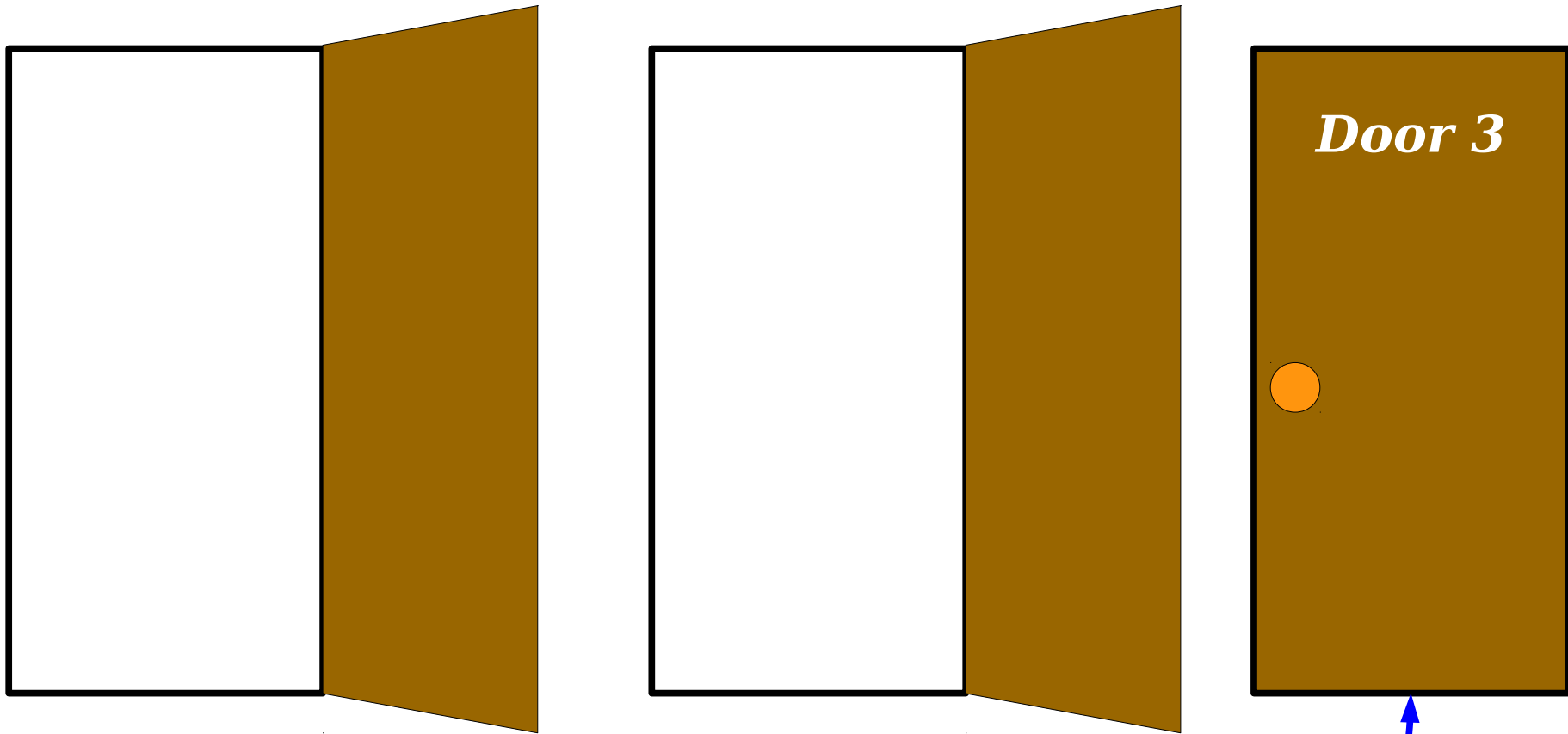
*There's something hidden behind one of these doors.
Which door is it hidden behind?*



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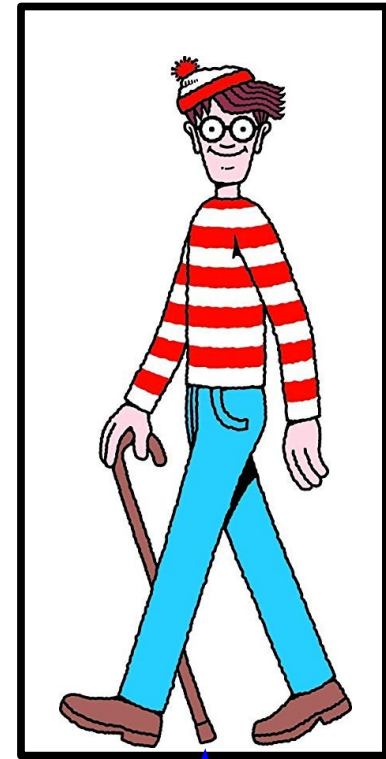
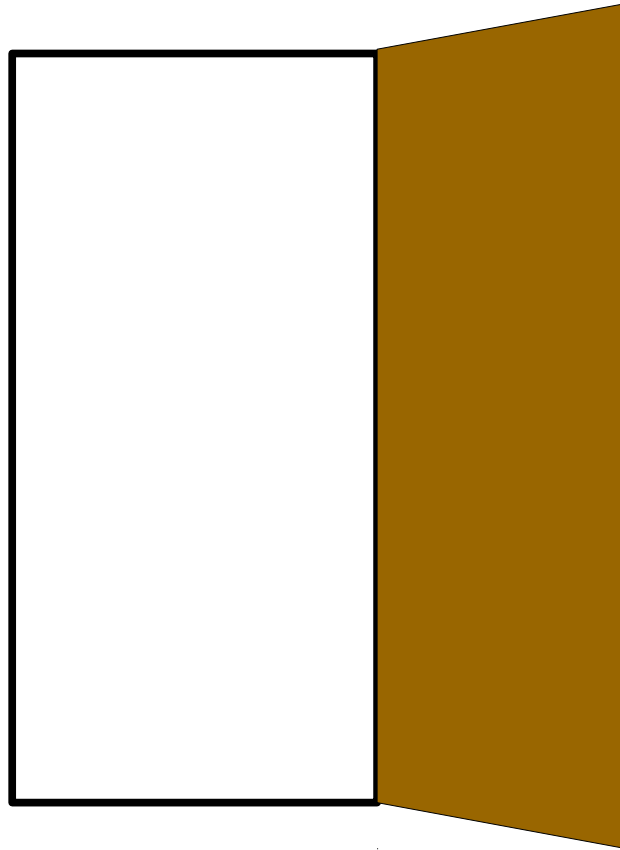
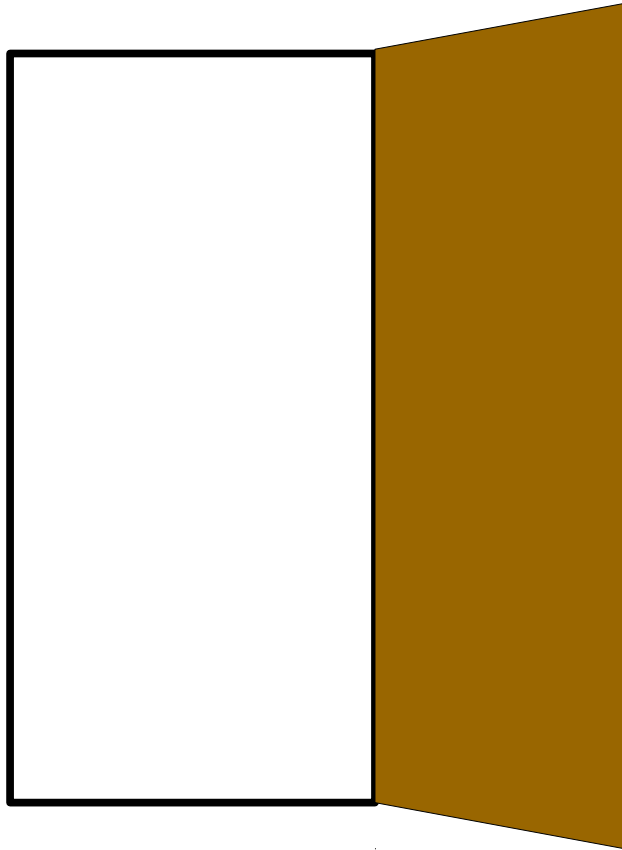


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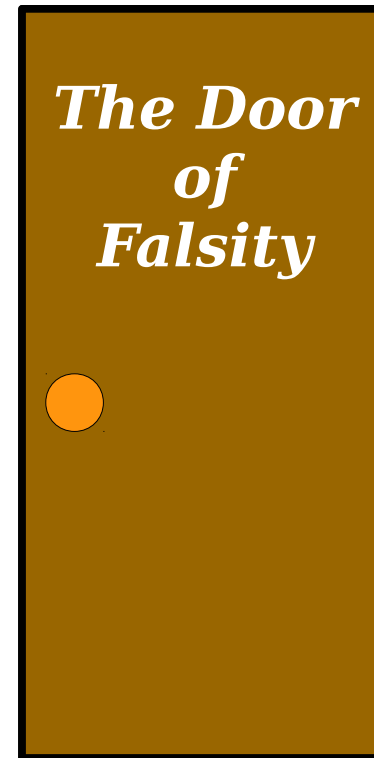
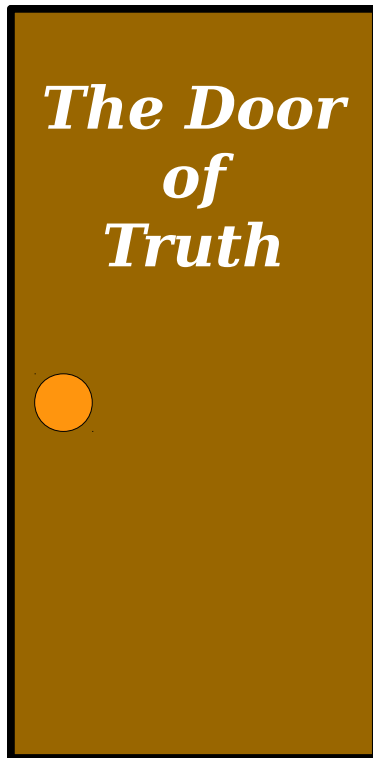
Even without opening this door, we know whatever is hidden has to be here.

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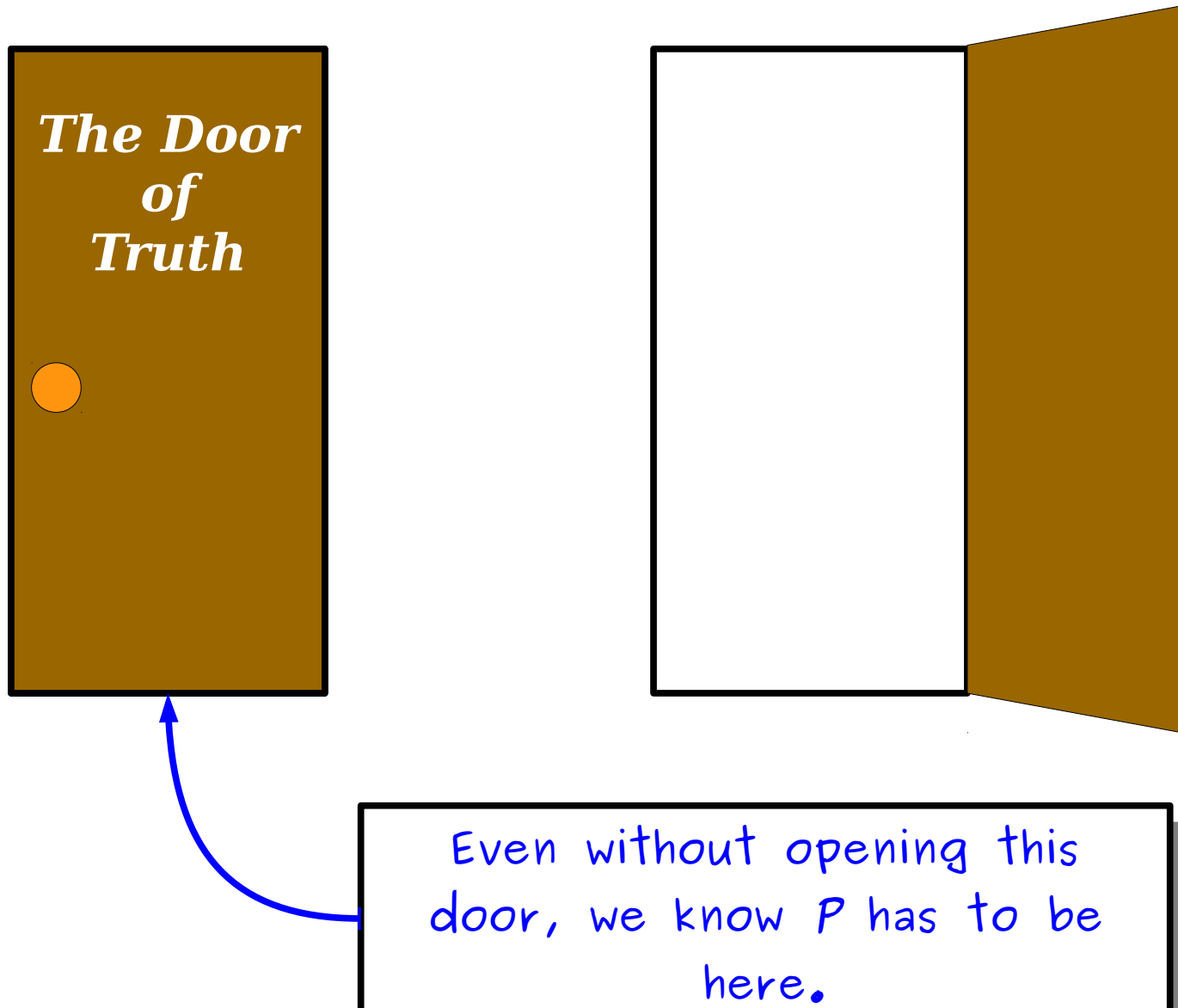


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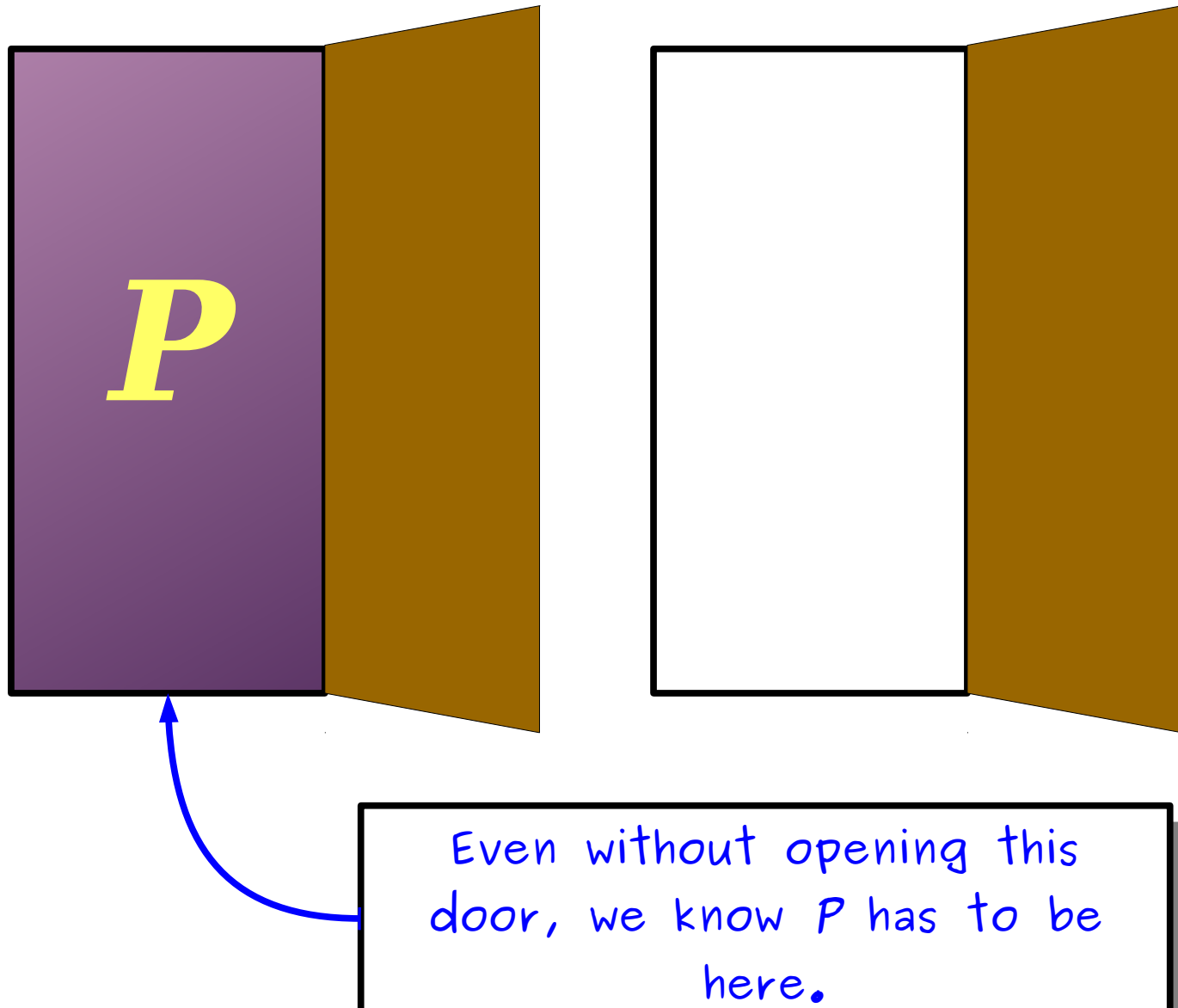
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If statement P is not false, what does that tell you?*



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A ***proof by contradiction*** shows that some statement P is true by showing that it cannot be false.

"When you have eliminated the impossible, whatever remains, however improbable, must be the truth."

- Sherlock Holmes



Proof by Contradiction

- To prove a statement P is true using a proof by contradiction, do the following:
 - Make the assumption that P is **false**.
 - Beginning with this assumption, use logical reasoning to conclude something that is clearly impossible.
 - For example, that $1 = 0$, that $x \in S$ and $x \notin S$, etc.
 - Conclude that P cannot be false, so P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
 - $|\emptyset| = 0$
 - $|\{1, 2, 3\}| = 3$
 - $|\{n \in \mathbb{N} \mid n < 137\}| = 137$
 - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- **Question:** Is there a “largest” set? That is, is there a set that's bigger than every other set?

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Notice that we're announcing

1. that this is a proof by contradiction,
- and
2. what, specifically, we're assuming.

This helps the reader understand where we're going. Remember - proofs are meant to be read by other people!

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

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Proving Implications

- To prove the implication

“If P is true, then Q is true.”
- you can use these three techniques:
 - ***Direct Proof.***
 - Assume P is true, then prove Q is true.
 - ***Proof by Contrapositive.***
 - Assume Q is false, then prove that P is false.
 - ***Proof by Contradiction.***
 - ... what does this look like?

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What is the negation of our theorem?

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Recap: Negating Implications

- To prove the statement

“For any x , if $P(x)$ is true, then $Q(x)$ is true”

by contradiction, we do the following:

- Assume this entire purple statement is false.
 - Derive a contradiction.
 - Conclude that the statement is true.
- What is the negation of the above purple statement?

**“There is an x where
 $P(x)$ is true and $Q(x)$ is false”**

<i>If you want to prove this by contradiction...</i>	<i>...assume this.</i>
All <i>P</i> 's are <i>Q</i> 's.	Some <i>P</i> is not a <i>Q</i> .
No <i>P</i> 's are <i>Q</i> 's.	Some <i>P</i> is a <i>Q</i> .
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Some <i>P</i> is not a <i>Q</i> .	All <i>P</i> 's are <i>Q</i> 's.
If <i>P</i> is true, then <i>Q</i> is true.	<i>P</i> is true, but <i>Q</i> is not true.
<i>P</i> is true and <i>Q</i> is true.	<i>P</i> is false, or <i>Q</i> is false, or both are false.
<i>P</i> is true or <i>Q</i> is true, or both are true.	<i>P</i> is false and <i>Q</i> is false.

What We Learned

- ***What's an implication?***

- It's a statement of the form “if P , then Q ,” and states that if P is true, then Q is true.

- ***How do you negate formulas?***

- It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.

- ***What is a proof by contrapositive?***

- It's a proof of an implication that instead proves its contrapositive.
- (The contrapositive of “if P , then Q ” is “if not Q , then not P .”)

- ***What's a proof by contradiction?***

- It's a proof of a statement P that works by showing that P cannot be false.

Next Time

- ***Mathematical Logic***
 - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
 - Reasoning about simple statements.
- ***Propositional Equivalences***
 - Simplifying complex statements.